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A Compact Finite Difference Scheme on a Non-equidistant Mesh

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A fourth-order compact finite difference scheme is presented for second-order partial differential equations. The scheme is derived for a one-dimensional non-equidistant mesh, which makes it particularly useful in problems with sharp boundary layers. The inclusion of general boundary conditions does not reduce the order of the scheme for the boundary points. The scheme is tested on a representative model equation; we shortly discuss its stability in the simplest time-dependent heat flow problem and its use in more dimensions.

1. INTRODUCTION

In this paper we present a fourth-order compact finite difference scheme for a set of second-order ordinary or partial differential equations. The scheme is based on a non-equidistant mesh, which makes it particularly useful in problems involving sharp boundary layers. We have used the scheme, for example, in calculations of steady-state density profiles of several ion species in fusion plasmas [1].

The derivation of the purely tridiagonal scheme, including boundary conditions, is presented for a one-dimensional non-equidistant mesh. It is based on a method attributable to Krause [2]; the elimination procedures required were partly performed by means of an algebraic manipulation program. Results of tests on a representative model equation are given.

Generalization of the scheme in more dimensions, using ADI-type schemes, is straightforward. For one-dimensional problems such as

$$c_1 y'' + c_2 y' + c_3 y = d \tag{1}$$

with coefficients and right-hand sides that are computed with few arithmetic operations, the fourth-order scheme requires approximately the same computational effort as a second-order scheme for the same accuracy. In most plasma-physics' problems, the coefficients and right-hand sides are obtained from lengthy

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calculations and here the use of the fourth-order scheme pays because of the reduction of the required number of mesh points.

The stability of the scheme in time-dependent problems is shortly discussed; a stability analysis is given for the simplest heat flow equation.

2. The Compact Scheme

The basic idea behind compact schemes is that one supplements the second-order differential equation with fourth-order relations between the function values y and the spatial first and second derivatives, $F(=y')$ and $S(=y'')$, on three adjacent mesh points. In this way, a scheme of fourth order, yet based on only three points, is constructed [3]. The required fourth-order relations can be obtained in various ways, e.g., from Taylor expansions as we shall show later in this chapter. For a review of compact methods we refer to Hirsh [4].

The most straightforward method in which one adds two relations to the differential equation at each inner mesh point and solves the resulting block tridiagonal system for y, F, and S, has the disadvantage that it requires a boundary condition on the second derivative [3, 4]. In order to avoid the use of points outside the computational domain, this boundary condition must be based on only two points, which at best is of third order [4].

A different approach, attributable to Krause [2, 4], does not have this advantage. Here, a system of fourth-order relations together with the differential equation on three adjacent mesh points is formulated such that the derivates F and S can be eliminated. Just one equation for the three function values remains, and one ends up with a purely tridiagonal system. A proper combination of fourth-order relations with both the boundary conditions and the differential equation ensures a fourth-order truncation error also at the boundaries.

We have used the same method as Krause, but derived the scheme for a non-equidistant mesh, which makes the reduction of the system of equations to a tridiagonal scheme more involved. We used the algebraic manipulation program SCHOONSCHIP [5] to perform the major part of this reduction.

Next, we shall give the fourth-order relations we used as a supplement to the differential equation, discuss the reduction of the system to a tridiagonal one, present the scheme, and show the inclusion of general boundary conditions on the function and its first derivative.

From the Taylor series expansion for the function values y at points $j+1$ and $j-1$ on the mesh x_j,

$$y^+ - y^0 \approx \Delta_+ F^0 + \tfrac{1}{2}\Delta_+^2 S^0 + \tfrac{1}{6}\Delta_+^3 y^{\text{iii}} + \tfrac{1}{24}\Delta_+^4 y^{\text{iv}} + \tfrac{1}{120}\Delta_+^5 y^{\text{v}} + \tfrac{1}{720}\Delta_+^6 y^{\text{vi}}, \quad (2)$$

$$y^0 - y^- \approx \Delta_- F^0 - \tfrac{1}{2}\Delta_-^2 S^0 + \tfrac{1}{6}\Delta_-^3 y^{\text{iii}} - \tfrac{1}{24}\Delta_-^4 y^{\text{iv}} + \tfrac{1}{120}\Delta_-^5 y^{\text{v}} - \tfrac{1}{720}\Delta_-^6 y^{\text{vi}}, \quad (3)$$

we obtain

$$\Delta_- y^+ - \Delta_t y^0 + \Delta_+ y^- \approx \tfrac{1}{2}\Delta_+\Delta_-\Delta_t S^0 + \tfrac{1}{6}\Delta_+\Delta_-(\Delta_+^2 - \Delta_-^2)\, y^{\text{iii}} + \tfrac{1}{24}\Delta_+\Delta_-(\Delta_+^3 + \Delta_-^3)\, y^{\text{iv}} + \tfrac{1}{120}\Delta_+\Delta_-(\Delta_+^4 - \Delta_-^4)\, y^{\text{v}} + \tfrac{1}{720}\Delta_+\Delta_-(\Delta_+^5 + \Delta_-^5)\, y^{\text{vi}}, \tag{4}$$

$$\Delta_-^2 y^+ + (\Delta_+^2 - \Delta_-^2)\, y^0 - \Delta_+^2 y^- \approx \Delta_+\Delta_-\Delta_t F^0 + \tfrac{1}{6}\Delta_+^2\Delta_-^2\Delta_t\, y^{\text{iii}} + \tfrac{1}{24}\Delta_+^2\Delta_-^2(\Delta_+^2 - \Delta_-^2)\, y^{\text{iv}} + \tfrac{1}{120}\Delta_+^2\Delta_-^2(\Delta_+^3 + \Delta_-^3)\, y^{\text{v}} + \tfrac{1}{720}\Delta_+^2\Delta_-^2(\Delta_+^4 - \Delta_-^4)\, y^{\text{vi}}. \tag{5}$$

Here, $F \equiv y'$, $S \equiv y''$, superscripts $+$, 0, $-$ indicate points $j+1, j$, and $j-1$; $\Delta_+ \equiv x^+ - x^0$, $\Delta_- \equiv x^0 - x^-$, and $\Delta_t \equiv \Delta_+ + \Delta_-$.

Equations (4) and (5) form the basis for the derivation of fourth-order relations between $y^{+,0,-}$, $F^{+,0,-}$, and $S^{+,0,-}$.

Differentiation of both (4) and (5) once and twice yields

$$\Delta_- F^+ - \Delta_t F^0 + \Delta_+ F^- \approx \tfrac{1}{2}\Delta_+\Delta_-\Delta_t\, y^{\text{iii}} + \tfrac{1}{6}\Delta_+\Delta_-(\Delta_+^2 - \Delta_-^2)\, y^{\text{iv}} + \tfrac{1}{24}\Delta_+\Delta_-(\Delta_+^3 + \Delta_-^3)\, y^{\text{v}} + \tfrac{1}{120}\Delta_+\Delta_-(\Delta_+^4 - \Delta_-^4)\, y^{\text{vi}}, \tag{4a}$$

$$\Delta_- S^+ - \Delta_t S^0 + \Delta_+ S^- \approx \tfrac{1}{2}\Delta_+\Delta_-\Delta_t\, y^{\text{iv}} + \tfrac{1}{6}\Delta_+\Delta_-(\Delta_+^2 - \Delta_-^2)\, y^{\text{v}} + \tfrac{1}{24}\Delta_+\Delta_-(\Delta_+^3 + \Delta_-^3)\, y^{\text{vi}}, \tag{4b}$$

$$\Delta_-^2 F^+ + (\Delta_+^2 - \Delta_-^2) F^0 - \Delta_+^2 F^- \approx \Delta_+\Delta_-\Delta_t S^0 + \tfrac{1}{6}\Delta_+^2\Delta_-^2\Delta_t\, y^{\text{iv}} + \tfrac{1}{24}\Delta_+^2\Delta_-^2(\Delta_+^2 - \Delta_-^2)\, y^{\text{v}} + \tfrac{1}{120}\Delta_+^2\Delta_-^2(\Delta_+^3 + \Delta_-^3)\, y^{\text{vi}}, \tag{5a}$$

$$\Delta_-^2 S^+ + (\Delta_+^2 - \Delta_-^2) S^0 - \Delta_+^2 S^- \approx \Delta_+\Delta_-\Delta_t\, y^{\text{iii}} + \tfrac{1}{6}\Delta_+^2\Delta_-^2\Delta_t\, y^{\text{v}} + \tfrac{1}{24}\Delta_+^2\Delta_-^2(\Delta_+^2 - \Delta_-^2)\, y^{\text{vi}}. \tag{5b}$$

The substitution of y^{iii} from (5b) and of y^{iv} from (4b) into Eqs. (4), (4a), (5), and (5a) now yields

$$\Delta_- y^+ - \Delta_t y^0 + \Delta_+ y^- - \frac{\Delta_-}{12}(\Delta_+^2 + \Delta_+\Delta_- - \Delta_-^2)\, S^+ - \frac{\Delta_t}{12}(\Delta_+^2 + 3\Delta_+\Delta_- + \Delta_-^2) S^0 + \frac{\Delta_+}{12}(\Delta_+^2 - \Delta_+\Delta_- - \Delta_-^2)\, S^- \approx \frac{-\Delta_+\Delta_-\Delta_t}{1440}\{4(\Delta_+ - \Delta_-)(2\Delta_+^2 + 5\Delta_+\Delta_- + 2\Delta_-^2)\, y^{\text{v}} + (3\Delta_+^4 + 2\Delta_+^3\Delta_- - 7\Delta_+^2\Delta_-^2 + 2\Delta_+\Delta_-^3 + 3\Delta_-^4)\, y^{\text{vi}}\}, \tag{6}$$

$$\begin{aligned}
&\Delta_- F^+ - \Delta_t F^0 + \Delta_+ F^- - \frac{\Delta_-}{6}(\Delta_- + 2\Delta_+)\, S^+ \\
&\qquad - \frac{\Delta_t}{6}(\Delta_+ - \Delta_-)\, S^0 + \frac{\Delta_+}{6}(\Delta_+ + 2\Delta_-)\, S^- \\
&\qquad \approx \frac{-\Delta_+ \Delta_- \Delta_t}{720}\{10(\Delta_+^2 + \Delta_+\Delta_- + \Delta_-^2)\, y^{\mathrm{v}} \\
&\qquad + (\Delta_+ - \Delta_-)(4\Delta_+^2 + 5\Delta_+\Delta_- + 4\Delta_-^2)\, y^{\mathrm{vi}}\},
\end{aligned}$$

$$\begin{aligned}
&\Delta_-^2\, y^+ + (\Delta_+^2 - \Delta_-^2)\, y^0 - \Delta_+^2\, y^- - \Delta_+\Delta_-\Delta_t\, F^0 - \tfrac{1}{12}\Delta_+\Delta_-^2\Delta_t\, S^+ \\
&\qquad - \tfrac{1}{12}\Delta_+\Delta_-\Delta_t(\Delta_+ - \Delta_-)\, S^0 + \tfrac{1}{12}\Delta_+^2\Delta_-\Delta_t S^- \\
&\qquad \approx \frac{-\Delta_+^2\Delta_-^2\Delta_t}{1440}\{4(2\Delta_+^2 + 3\Delta_+\Delta_- + 2\Delta_-^2)\, y^{\mathrm{v}} \\
&\qquad + (\Delta_+ - \Delta_-)(3\Delta_+^2 + 5\Delta_+\Delta_- + 3\Delta_-^2)\, y^{\mathrm{vi}},
\end{aligned}$$

$$\begin{aligned}
&\Delta_-^2\, F^+ + (\Delta_+^2 - \Delta_-^2) F^0 - \Delta_+^2\, F^- - \tfrac{1}{3}\Delta_+\Delta_-^2\, S^+ - \tfrac{2}{3}\Delta_+\Delta_-\Delta_t S^0 - \tfrac{1}{3}\Delta_+^2\Delta_-\, S^- \\
&\qquad \approx \frac{-\Delta_+^2\Delta_-^2\Delta_t}{360}\{5(\Delta_+ - \Delta_-)\, y^{\mathrm{v}} + 2(\Delta_+^2 - \Delta_+\Delta_- + \Delta_-^2)\, y^{\mathrm{vi}}\}.
\end{aligned}$$

If we neglect the fourth-order truncation errors, Eqs. (6)–(9) supplemen ferential equation at the three points considered, i.e., the linear equations

$$c_3^k\, y^k + c_2^k\, F^k + c_1^k\, S^k = d^k \qquad (k = +, 0, -),$$

to form a system of seven equations in nine unknowns. It is now po eliminate F^k and S^k for $k = +, 0, -$ and end up with one equation unknowns, viz., the y^k. This equation is the required compact relation, th order tridiagonal scheme. We have used the algebraic manipulation SCHOONSCHIP [5] to perform the major part of the elimination proce

Finally, the following compact relation was obtained:

$$F_1\, y^+ + F_2\, y^0 + F_3\, y^- = R,$$

where

$$\begin{aligned}
F_k &= E_{1k}E_{24} - E_{2k}E_{14} && (k = 1, 2, 3),\\
E_{1k} &= D_{1k}D_{35} - D_{3k}D_{15} && (k = 1, 2, 3, 4),\\
E_{2k} &= D_{2k}D_{35} - D_{3k}D_{25} && (k = 1, 2, 3, 4),\\
R &= S_1 E_{24} - S_2 E_{14},
\end{aligned}$$

$$S_1 = T_1 D_{35} - T_3 D_{15}, \tag{12e}$$

$$S_2 = T_2 D_{35} - T_3 D_{25}, \tag{12f}$$

and

$$D_{11} = -\Delta_-^3 \{2(6\Delta_+^2 + 6\Delta_+ \Delta_- + \Delta_-^2) c_1^+ + 2\Delta_t (2\Delta_+ + \Delta_-)\Delta_+ c_2^+ + \Delta_t^2 \Delta_+^2 c_3^+\}, \tag{13a}$$

$$D_{12} = -2\Delta_t^3 \{(6\Delta_+^2 - 3\Delta_+ \Delta_- - \Delta_-^2) c_1^+ + \Delta_t (\Delta_+ - \Delta_-)\Delta_+ c_2^+\}, \tag{13b}$$

$$D_{13} = 2\Delta_+^3 \{(6\Delta_+^2 + 15\Delta_+ \Delta_- + 8\Delta_-^2) c_1^+ + \Delta_t (\Delta_+ + 2\Delta_-)\Delta_+ c_2^+\}, \tag{13c}$$

$$D_{14} = \Delta_t^3 \Delta_+ \Delta_- \{2(3\Delta_+ + \Delta_-) c_1^+ + \Delta_t \Delta_+ c_2^+\}, \tag{13d}$$

$$D_{15} = \Delta_t \Delta_+^3 \Delta_- \{2(3\Delta_+ + 2\Delta_-) c_1^+ + \Delta_t \Delta_+ c_2^+\}, \tag{13e}$$

$$T_1 = -\Delta_t^2 \Delta_+^2 \Delta_-^3 d^+, \tag{13f}$$

$$D_{21} = 2\Delta_-^4 c_1^0, \tag{14a}$$

$$D_{22} = \Delta_t^2 \{-2(3\Delta_+^2 - 2\Delta_+ \Delta_- + \Delta_-^2) c_1^0 + \Delta_+^2 \Delta_-^2 c_3^0\}, \tag{14b}$$

$$D_{23} = 2(3\Delta_+ + 4\Delta_-)\Delta_+^3 c_1^0, \tag{14c}$$

$$D_{24} = \Delta_t \Delta_+ \Delta_- \{2(2\Delta_+^2 + \Delta_+ \Delta_- - \Delta_-^2) c_1^0 + \Delta_t \Delta_+ \Delta_- c_2^0\}, \tag{14d}$$

$$D_{25} = 2\Delta_t \Delta_+^3 \Delta_- c_1^0, \tag{14e}$$

$$T_2 = \Delta_t^2 \Delta_+^2 \Delta_-^2 d^0, \tag{14f}$$

$$D_{31} = -2\Delta_-^4 c_1^-, \tag{15a}$$

$$D_{32} = -2\Delta_t^3 (3\Delta_+ - \Delta_-) c_1^-, \tag{15b}$$

$$D_{33} = \Delta_+^2 \{2(3\Delta_+^2 + 8\Delta_+ \Delta_- + 6\Delta_-^2) c_1^- - \Delta_t^2 \Delta_-^2 c_3^-\}, \tag{15c}$$

$$D_{34} = 2\Delta_t^3 \Delta_+ \Delta_- c_1^-, \tag{15d}$$

$$D_{35} = \Delta_t \Delta_+^2 \Delta_- \{2(2\Delta_+ + 3\Delta_-) c_1^- - \Delta_- \Delta_t c_2^-\}, \tag{15e}$$

$$T_3 = -\Delta_t^2 \Delta_+^2 \Delta_-^2 d^-. \tag{15f}$$

General boundary conditions

$$\beta y + \gamma F = \delta \tag{16}$$

are easily inserted. We add the linear equation in y and F representing the boundary condition to the system of seven equations (6)–(10c) for the first interior point away from the boundary. Thus, we have eight equations in nine unknows. Processing the elimination procedure now not only for $F^{+,0,-}$ and $S^{+,0,-}$, but also for y^- (right boundary) or y^+ (left boundary) yields one equation in two unknown function values, y_{N-1} and y_N respectively y_1 and y_2. This completes the tridiagonal system with fourth-order boundary conditions on both sides. The coefficients resulting from the elimination procedure at the boundaries are as follows.

At the left boundary

$$H_3 y_1 + H_2 y_2 = P, \tag{17}$$

with

$$H_k = F_1 G_k - F_k G_1, \tag{17a}$$

$$P = F_1 Q - G_1 R, \tag{17b}$$

$$G_k = \gamma(D_{34} E_{2k} - E_{24} D_{3k}) \qquad (k = 1, 2), \tag{17c}$$

$$G_3 = \beta E_{24} D_{35} + \gamma(D_{34} E_{23} - D_{33} E_{24}), \tag{17d}$$

$$Q = \delta E_{24} D_{35} + \gamma(D_{34} S_2 - E_{24} T_3), \tag{17e}$$

where F_1, F_2, F_3, D_{ij}, E_{ij}, R, S_2, and T_3 are to be taken from the compact scheme (Eqs. (11)–(15)) at point $j = 2$.

At the right boundary

$$L_1 y_N + L_2 y_{N-1} = M, \tag{18}$$

with

$$L_k = K_k F_3 - K_3 F_k \qquad (k = 1, 2), \tag{18a}$$

$$M = F_3 O - K_3 R, \tag{18b}$$

$$K_k = E_{24} H_k - E_{2k} H_4 \qquad (k = 1, 2, 3), \tag{18c}$$

$$O = E_{24} P - H_4 S_2, \tag{18d}$$

$$H_k = G_k D_{35} - G_5 D_{3k} \qquad (k = 1, 2, 3, 4), \tag{18e}$$

$$P = D_{35} Q - G_5 T_3, \tag{18f}$$

$$G_1 = -\Delta_-^3 \Delta_+ \Delta_t \beta - 2\Delta_-^3 (2\Delta_+ + \Delta_-)\gamma, \tag{18g}$$

$$G_2 = -2\Delta_t^3 (\Delta_+ - \Delta_-)\gamma, \tag{18h}$$

$$G_3 = 2\Delta_+^3 (\Delta_+ + 2\Delta_-)\gamma, \tag{18i}$$

$$G_4 = \Delta_t^3 \Delta_+ \Delta_- \gamma, \tag{18j}$$

$$G_5 = \Delta_+^3 \Delta_- \Delta_t \gamma, \tag{18k}$$

$$Q = -\Delta_t \Delta_+ \Delta_-^3 \delta, \tag{18l}$$

where F_1, F_2, F_3, D_{ij}, E_{ij}, R, S_2, T_3, Δ_+, Δ_-, and Δ_t are to be taken from the compact scheme at point $j = N - 1$.

3. Test of the Scheme

As an example we have chosen a simple transport problem in which a particle source is prescribed as $\alpha x^k y$, where α is a constant and x is the spatial coordinate; the particle flux Γ is coupled to the gradient of the density y by a constant diffusion coefficient D. Thus, in a plane geometry we have

$$\frac{d\Gamma}{dx} = \alpha x^k y, \tag{19a}$$

$$\Gamma = -D \frac{dy}{dx}. \tag{19b}$$

These equations combine to the second-order ODE

$$y'' + a x^k y = 0, \tag{19c}$$

with $a = \alpha/D$, which has the analytical solution [6]

$$y(x) = C_1 \sqrt{x}\, J_{1/p}\left(\frac{2\sqrt{a}}{p}\, x^{p/2}\right) + C_2 \sqrt{x}\, J_{-1/p}\left(\frac{2\sqrt{a}}{p}\, x^{p/2}\right), \tag{19d}$$

where $J_{1/p}$ is the Bessel function of fractional order $1/p$, $p = k + 2$, and $C_{1,2}$ are the integration constants. Insertion of boundary conditions $y(1) = y_w$ and $y'(0) = 0$ and taking a power series representation [6] leads to

$$y(x) = \frac{y_w}{\sum_{j=0}^{\infty} g_j} \sum_{j=0}^{\infty} g_j x^{(k+2)j} \tag{19e}$$

with

$$g_j = \frac{(-1)^j}{j!\, \Gamma(1 + j - 1/(k+2))} \left(\frac{\sqrt{a}}{k+2}\right)^{2j - 1/(k+2)} \tag{19f}$$

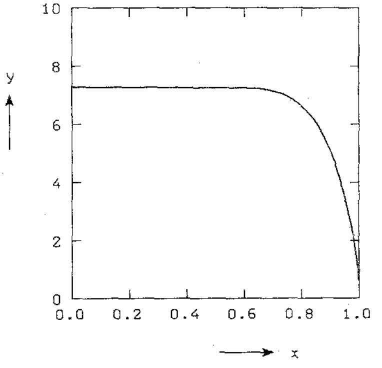

FIG. 1. The solution of the differential equation: $y'' + ax^k y = 0$ for $a = 150$ and $k = 10$.

Figure 1 shows the solution of this model equation for $a = 150$ and $k = 10$. The coefficients a and k allow for tuning of the width of the boundary layer and of the value of the function at $x = 0$.

The meshes we used for numerically solving Eq. (19c) are given by

$$x_j = 1 - \left\{1 - \frac{j-1}{N-1}\right\}^m, \qquad \text{for } m = 1, 2, 3, 4, \ j = 1, 2, \ldots, N, \tag{20}$$

which enables us to take meshes from equidistant ($m = 1$) to meshes with a high density of points in the boundary layer.

The numerical results for $a = 150$ and $k = 10$ are shown in Fig. 2, where the maximum absolute error $\varepsilon = \mathrm{Max}_j |y_j - y(x_j)|$, with y_j the numerical and $y(x_j)$ the analytical solution at x_j, is plotted vs the average mesh spacing, $1/(N-1)$. For comparison, the same quantity is also plotted for the usual 3-point second-order scheme, where

$$y'' = \frac{2(\Delta_- y^+ - \Delta_t y^0 + \Delta_+ y^-)}{\Delta_+ \Delta_- \Delta_t} - \frac{1}{3}(\Delta_+ - \Delta_-)\, y^{\mathrm{iii}} - \frac{1}{12}\frac{(\Delta_+^3 + \Delta_-^3)}{\Delta_t} y^{\mathrm{iv}}. \tag{21}$$

The integer values in Fig. 2 denote the value of m used for the mesh.

The behaviour of the error for the two schemes is as expected. The truncation error proportional to y^{iii} in the second derivative (Eq. (21)) of the second-order scheme is also of second order in $1/(N-1)$ as long as the function prescribing the density of the mesh points is smooth enough. Both schemes show an increase of the error when m is raised above 2. In that case, the density of points near $x = 0$

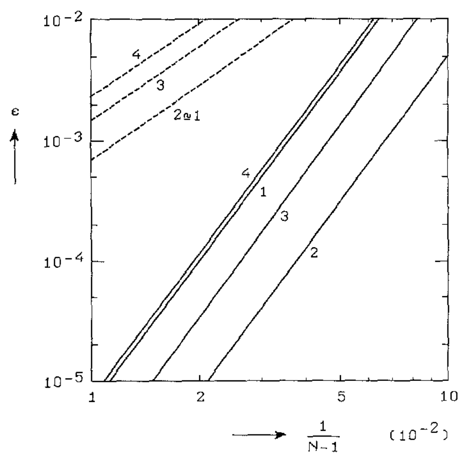

FIG. 2. The maximum error ε of the compact scheme (solid line) and of the second-order scheme (dashed line) for the model equation: $y'' + 150x^{10}y = 0$. Integer values denote the choice of m, the power in Eq. (20).

becomes too small and the error is no longer mainly determined by what happens in the boundary layer. The reduction of the error by a factor of 10, or equivalently of the number of required mesh points by a factor of 2, when changing from an equidistant ($m=1$) to a parabolic distribution ($m=2$) for the compact scheme, occurs also at larger values of the constants a and k.

For $m=2$, the second-order scheme roughly needs four to five times as many points as the compact scheme at a relative error level (measured by the euclidean norm of the error and function vectors) of 10^{-3}. At high values of the parameter a, even more points are required; the number of points is roughly proportional to $\sqrt{y(o)}$ for the second-order scheme and proportional to $\sqrt[4]{y(o)}$ for the compact scheme. This reflects the order of the schemes. An analysis of the computational effort, i.e., the number of multiplications, shows that the effort required to reach this accuracy is approximately the same for both schemes. However, it should be stressed again that in most practical problems the calculation of the coefficients of the differential equation is more involved. Then, the strong reduction of the required number of mesh points compensates by far the effort in the construction of the compact scheme.

4. Stability

The stability of the compact scheme in time-dependent problems can be analysed in the usual way with the von Neumann method [7]. The right-hand side R of Eq. (11) in this case is a linear combination of the time derivative at the mesh points considered. If we write the solution to the difference equation as

$$y_j^n = y(n\,\delta t, x_j) = \xi^n \exp(imx_j) \tag{22}$$

and require the multiplication factor $|\xi|$ to be less than 1, the restriction on the time step is obtained. In general, this restriction will be a complicated one. We performed the analysis for the heat flow problem

$$y_t = \sigma y'', \tag{23}$$

using forward differencing for the time derivative. The resulting restriction on the time step δt for a non-equidistant mesh reads

$$\frac{\sigma\delta t}{\Delta_t^2} \leqslant \frac{1}{6(1-2\upsilon)} \frac{(\alpha^2-\alpha+1)^2 - \alpha^2(1-\alpha)^2 \cos m\Delta_t - \alpha^2 \cos(1-\alpha)m\Delta_t - (1-\alpha)^2 \cos \alpha m\Delta_t}{\alpha^2-\alpha+1+\alpha(1-\alpha)\cos m\Delta_t - \alpha\cos(1-\alpha)m\Delta_t - (1-\alpha)\cos\alpha m\Delta_t}, \tag{24}$$

where $\alpha = \Delta_-/\Delta_t$ and υ is the implicitness factor [7]. This holds for $\upsilon < 0.5$; for

$\upsilon \geqslant 0.5$ the scheme is unconditionally stable. If we take the mesh to be equidistant ($\Delta = \Delta_t/2$), the restriction for a fully explicit scheme reduces to

$$\frac{\sigma\,\delta t}{\Delta^2} \leqslant \frac{1}{3}. \tag{24a}$$

The second-order scheme here yields the well-known upper value of 1/2. Thus, as long as the truncation error in the discretization of the time derivative can be neglected, the compact scheme allows for a larger time step than does the second-order scheme, because, for the same accuracy, Δ is allowed to be much larger in the former.

5. More-Dimensional Problems

The compact scheme can be used in more dimensions, at least in procedures where the spatial operator is made implicit successively for all dimensions as in ADI-type schemes. A 2-dimensional problem

$$u_t = L_x(u) + L_y(u), \tag{25}$$

where L_x and L_y are second-order spatial differential operators, for example, may be solved by means of the method of Peaceman and Rachford [8], which consists of two time steps, each of $1/2\delta t$:

$$u^{n+1/2} - u^n = \frac{2}{\delta t}\{L_x(u)\}^{n+1/2} + \frac{2}{\delta t}\{L_y(u)\}^n, \tag{26a}$$

$$u^n - u^{n+1/2} = \frac{2}{\delta t}\{L_x(u)\}^{n+1/2} + \frac{2}{\delta t}\{L_y(u)\}^{n+1}. \tag{26b}$$

L_x and L_y are then evaluated according to the compact scheme. Note that in the first step we need $L_y(u)$ at three x values, because the right-hand side of the compact scheme is a linear combination of u_t and $L_y(u)$ at the three x mesh points. The same holds in the second step for $L_x(u)$.

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